# INTERSECTION OF LINEARLY RECURRING SEQUENCES

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## Abstract

In this paper, we investigate the common terms of the Leonardo $\{Le_n\}_{n\geq 0}$ and Jacobsthal $\{J_m\}_{m\geq 0}$ sequences. To accomplish this, we use Baker's theory of linear forms in logarithms of algebraic numbers along with a variation of he Baker-Davenport reduction method solve the Diophantine equation $Le_n = J_m$.

## Previous Work

In 1978, Mignotte (see [5]) showed that, assuming certain weak technical assumptions, the intersection of two linearly recurring sequences contains only a finite number of terms. Alekseyev explains how to find the intersection of two Lucas sequences of the forms $U_n(P, \pm 1)$ or $V_n(P, \pm 1)$ with $P \in \mathbb{Z}$. Szalay provided an approach to calculate the intersection of the binary sequences with arbitrary initial conditions.
Catarino in ([1], Lemma 2.1) and Horadam in ([3]) show that each term in the Leonardo and Jacobsthal sequences, respectively, is an odd number; this, along with several papers, studied the problem of finding the common terms of some interesting integer sequences.

## Our result

Motivated by the mentioned above, we consider the following equation

$$Le_n = J_m. \tag{1}$$

We prove the following result.

**Theorem**

All solutions of the Diophantine equation $Le_n = J_m$ in non-negative integers $(n, m)$ with $n \geq 2$ and $m \geq 2$ are given by

$$(n, m) \in \{(2, 3), (3, 4)\}.$$

## Before starting

**What is a Diophantine équation?**

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial with integer coefficients. A **Diophantine equation** is an equation of the form:

$$P(x_1, x_2, \ldots, x_n) = 0$$

where the solution set is restricted to the ring of integers:

$$(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$$

## Preliminary results

In order to prove our main result, we use a Baker-type lower bound for a non-zero linear forms in logarithms of algebraic numbers. We state a result of Matveev about the general lower bound for linear forms in logarithms

**Matveev's theorem**

Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree $D$, $b_1, \ldots, b_t$ are rational integers. Put

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

and

$$B \geq \max\{|b_1|, \ldots |b_t|\}.$$

Let $A_1, A_2, ..., A_t$ be positive real numbers such that

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \text{ for all } i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| \geq \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\right).$$

Another result which will play an important role in our proof is due to Dujella and Pethö [2, Lemma 5 (a)].

**Dujella and Pethő Lemma**

Let $M$ be a positive integer, let $p/q$ be a convergent of the continued fraction of the irrational $\gamma$ such that $q > 6M$, and let $A, B, \mu$ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers $u$, $v$ and $w$ with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

## Results and proofs

**The First computations**

Since $Le_0 = Le_1 = J_1 = 1$, we will assume $n \geq 2$ and $m \geq 2$ to avoid the trivial solutions. Using *Mathematica*, we obtain all the solutions of equation (1) in the ranges $n \in [2, 100]$ and $m \in [2, 200]$ as listed in our Theorem. From now, assume that $m > 100$.

### *Lemma*

*All solutions of* (1) *satisfy*

$$(m-2)\frac{\log 2}{\log \alpha} - 1 < n < (m-1)\frac{\log 2}{\log \alpha} < 2m.$$

## Bounding the variables.

In this subsection, we use Matveev's theorem to achieve our goal of finding an upper bounds for n and m. we get

$$m < 5.9 \cdot 10^{14}.$$

From Lemma 1 once again, we deduce that

$$n < 1.18 \cdot 10^{15}.$$

## Reducing the bounds

To reduce the bounds, we definitely need to use the Dujella and Pethő Lemma. We get

$$m \leq \frac{\log(18 \cdot q_{37}/0.12)}{\log 2} < 64.$$

Which contradicts our assumption that $m > 100$. Hence, we have no more solutions for Eq. (1) in the range $m > 100$.

## References

[1] P.M. Catarino and A. Borges, On Leonardo numbers, *Acta Math. Univ. Comen.* **89** (1) (2019), 75–86.

[2] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* **49** (1998), 291–306.

[3] A. F. Horadam, Jacobsthal Representation Numbers. *Fib. Quart.* **34** (1996), 40–54.

[4] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180. Translation in *Izv. Math.* **64** (2000), 1217–1269.

[5] M. Mignotte, Intersection des images de certaines suites recurrentes lineaires, *Theor. Comput.* **7**(1978), 117–121.